

Arrangement of hyperplanes II: Szenes formula and Eisenstein series

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Dedicated to Victor Guillemin, for his 60th birthday

Abstract: The aim of this article is to generalize in several variables some formulae for Eisenstein series in one variable. In particular, we relate Szenes formula to Eisenstein series and we give another proof of it.

1 Introduction

Consider a sequence $(\alpha_1, \alpha_2, \dots, \alpha_k)$ of linear forms in r complex variables, with integral coefficients. The linear forms α_j need not be distinct. For example $r = 2$ and $\alpha_1 = \alpha_2 = z_1$, $\alpha_3 = \alpha_4 = z_2$, $\alpha_5 = \alpha_6 = z_1 + z_2$. For any such sequence, Zagier [5] introduced the series

$$\sum_{n \in \mathbb{Z}^r, \langle \alpha_j, n \rangle \neq 0} \frac{1}{\prod_{j=1}^k \langle \alpha_j, n \rangle}.$$

Assuming convergence, its sum is a rational multiple of π^k . For example [5], we have

$$\sum_{n_1 \neq 0, n_2 \neq 0, n_1 + n_2 \neq 0} \frac{1}{n_1^2 n_2^2 (n_1 + n_2)^2} = \frac{(2\pi)^6}{30240}.$$

These numbers are natural multidimensional generalizations of the value of the Riemann zeta function at even integers. A. Szenes gave in ([3], Theorem 4.4) a residue formula for these numbers, relating them to Bernoulli numbers. The formula of Szenes [3] is the multidimensional analogue of the residue formula

$$\sum_{n \neq 0} \frac{1}{n^{2l}} = (2\pi)^{2l} \frac{B_{2l}}{(2l)!} = (-1)^l (2\pi)^{2l} \operatorname{Res}_{z=0} \left(\frac{1}{z^{2l}(1-e^z)} \right).$$

A motivation for computing such sums comes from the work of E. Witten [4]: In the special case where α_j are the positive roots of a compact connected Lie group G , each of these roots being repeated with multiplicity $2g - 2$, Witten expressed the symplectic volume of the space of homomorphisms of the fundamental group of a Riemann surface of genus g into G , in terms of these sums. In [2], L. Jeffrey and F. Kirwan proved a special case of Szenes formula leading to the explicit computation of this symplectic volume, when G is $SU(n)$.

Our interest in such series comes from a different motivation. Let us consider first the one-dimensional case. By Poisson formula, the convergent series, for $Re(z) > 0$, $\sum_{m=1}^{\infty} me^{-mz}$ is also equal to $\sum_{n \in \mathbb{Z}} 1/(z + 2i\pi n)^2$. Similarly, sums of products of polynomial functions with exponential functions over all integral points of a r -dimensional rational convex cone are related to functions of r complex variables of the form

$$\psi(z) = \sum_{n \in \mathbb{Z}^r} \frac{1}{\prod_{j=1}^k \langle \alpha_j, z + 2i\pi n \rangle}.$$

When this series is not convergent, introduce the oscillating factor $e^{\langle t, 2i\pi n \rangle}$ and define the Eisenstein series

$$\psi(t, z) = \sum_{n \in \mathbb{Z}^r} \frac{e^{\langle t, z + 2i\pi n \rangle}}{\prod_{j=1}^k \langle \alpha_j, z + 2i\pi n \rangle},$$

a generalized function of $t \in \mathbb{R}^n$.

In Section 3, we construct a decomposition of an open dense subset of \mathbb{R}^n into alcoves such that $t \mapsto \psi(t, z)$ is given on each alcove by a polynomial in t , with coefficients rational functions of e^z . Our first theorem (Theorem 19) gives an explicit residue formula for $\psi(t, z)$. It follows easily from the obvious behaviour of $\psi(t, z)$ under differentiation in z .

This formula allows to give a residual meaning “ $\psi(t, 0)$ ” for the value of $\psi(t, z)$ at $z = 0$, although $\psi(t, z)$ has clearly poles along all hyperplanes $\langle \alpha_j, z \rangle = 0$. An alternative way to define $\psi(t, 0)$ is to remove all infinities $1/\alpha_j$ in the series

$$\psi(t, 0) = \sum_{n \in \mathbb{Z}^r} \frac{e^{\langle t, 2i\pi n \rangle}}{\prod_{j=1}^k \langle \alpha_j, 2i\pi n \rangle}.$$

Indeed, we prove that the residue formula for “ $\psi(t, 0)$ ” coincides with the

renormalized sum :

$$“\psi(t, 0)” = \sum_{n \in \mathbb{Z}^r, \langle \alpha_j, n \rangle \neq 0} \frac{e^{\langle t, 2i\pi n \rangle}}{\prod_{j=1}^k \langle \alpha_j, 2i\pi n \rangle}.$$

This equality gives another proof of Szenes residue formula, as a “limit” of a natural formula for $\psi(t, z)$ when $z \rightarrow 0$ along a generic line.

To illustrate our method, let us consider the one-dimensional case. For $k \geq 2$, we can define the Eisenstein series

$$E_k(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z + 2i\pi n)^k}.$$

Clearly, $E_k(z)$ is periodic in z with respect to translation by the lattice $2i\pi\mathbb{Z}$. From the residue theorem, when y is not in $2i\pi\mathbb{Z}$, we have the kernel formula:

$$(1) \quad E_k(y) = \text{Res}_{z=0} \left(\frac{1}{z^k (1 - e^{z-y})} \right).$$

Observe that the right-hand side has a meaning when $y = 0$, and equals by definition the Bernoulli number $B_k/k!$. The function

$$E_k(y) = \frac{1}{y^k} + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{(y + 2i\pi n)^k}$$

has a Laurent expansion at $y = 0$, with $1/y^k$ as Laurent negative part. We see from the residue formula that the constant term $CT(E_k) = \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{(2i\pi n)^k}$ equals $\text{Res}_{z=0} \left(\frac{1}{z^k (1 - e^z)} \right)$.

In view of this example, we call the value “ $\psi(t, 0)$ ” of $\psi(t, y)$ at $y = 0$ the constant term of the Eisenstein series

$$\sum_{n \in \mathbb{Z}^r} \frac{e^{\langle t, z + 2i\pi n \rangle}}{\prod_{j=1}^k \langle \alpha_j, z + 2i\pi n \rangle}.$$

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2 Kernel formula

In this section, we recall briefly results of [1] with slightly modified notation. Let V be a r -dimensional complex vector space. Let V^* be the dual vector

space and let $\Delta \subset V^*$ be a finite subset of non-zero linear forms. Each $\alpha \in \Delta$ determines a hyperplane $\{\alpha = 0\}$ in V . Consider the hyperplane arrangement

$$\mathcal{H} = \bigcup_{\alpha \in \Delta} \{\alpha = 0\}.$$

An element $z \in V$ is called **regular** if z is not in \mathcal{H} . If S is a subset of V , we write S_{reg} for the set of regular elements in S . The ring R_Δ of rational functions with poles on \mathcal{H} is the ring $\Delta^{-1}S(V^*)$ generated by the ring $S(V^*)$ of polynomial functions on V , together with inverses of the linear functions $\alpha \in \Delta$. The ring R_Δ has a \mathbb{Z} -gradation by the homogeneous degree which can be positive or negative. Elements of R_Δ are defined on the open subset V_{reg} . (Our notation differs from [1] in that the roles of V and V^* are interchanged.)

In the one variable case, the function $1/z$ is the unique function which cannot be obtained as a derivative. There is a similar description of a complement space to the space of derivatives in the ring R_Δ that we recall now.

A subset σ of Δ is called a **basis of Δ** , if the elements $\alpha \in \sigma$ form a basis of V . We denote by $\mathcal{B}(\Delta)$ the set of bases of Δ . An **ordered basis** is a sequence $(\alpha_1, \alpha_2, \dots, \alpha_r)$ of elements of Δ such that the underlying set is a basis. We denote by $OB(\Delta)$ the set of ordered bases.

For $\sigma \in \mathcal{B}(\Delta)$, set

$$\phi_\sigma(z) := \frac{1}{\prod_{\alpha \in \sigma} \alpha(z)}.$$

We call ϕ_σ a **simple fraction**. Setting $z_j = \langle z, \alpha_j \rangle$, we have

$$\phi_\sigma(z) = \frac{1}{z_1 z_2 \cdots z_r}.$$

Definition 1 *The subspace S_Δ of R_Δ spanned by the elements ϕ_σ , $\sigma \in \mathcal{B}(\Delta)$, will be called the space of **simple elements** of R_Δ :*

$$S_\Delta = \sum_{\sigma \in \mathcal{B}(\Delta)} \mathbb{C} \phi_\sigma.$$

The space S_Δ consists of homogeneous rational functions of degree $-r$. However, not every homogeneous element of degree $-r$ of R_Δ is in S_Δ (for example, in the preceding notation if $r \geq 2$, both functions $1/z_1^r$ and z_2/z_1^{r+1}

are not in S_Δ). Furthermore we must be careful, as the elements ϕ_σ may be linearly dependent: for example, if $V = \mathbb{C}^2$ and $\Delta = \{z_1, z_2, z_1 + z_2\}$, we have

$$S_\Delta = \mathbb{C} \frac{1}{z_1 z_2} + \mathbb{C} \frac{1}{z_1(z_1 + z_2)} + \mathbb{C} \frac{1}{z_2(z_1 + z_2)}$$

and we have the relation

$$\frac{1}{z_1 z_2} = \frac{1}{z_1(z_1 + z_2)} + \frac{1}{z_2(z_1 + z_2)}.$$

A description due to Orlik and Solomon of all linear relations between the elements ϕ_σ is given in [1] Proposition 13.

Definition 2 *A basis B of $\mathcal{B}(\Delta)$ is a subset of $\mathcal{B}(\Delta)$ such that the elements $(\phi_\sigma, \sigma \in B)$, form a basis of S_Δ :*

$$S_\Delta = \bigoplus_{\sigma \in B} \mathbb{C} \phi_\sigma.$$

We let elements v of V act on R_Δ by differentiation:

$$(\partial(v)f)(z) := \frac{d}{d\epsilon} f(z + \epsilon v)|_{\epsilon=0}.$$

Then the following holds ([1], Proposition 7.)

Theorem 3

$$R_\Delta = \partial(V)R_\Delta \oplus S_\Delta.$$

Thus we see that only simple fractions cannot be obtained as derivatives.

As a corollary of this decomposition, we can define the projection map

$$Res_\Delta : R_\Delta \rightarrow S_\Delta.$$

The projection $Res_\Delta f(z)$ of a function $f(z)$ is a function of z that we called the **Jeffrey-Kirwan residue** of f . By definition, this function can be expressed as a linear combination of the simple fractions ϕ_σ . The main property of the map Res_Δ is that it vanishes on derivatives, so that for $v \in V$, $f, g \in R_\Delta$:

$$(2) \quad Res_\Delta ((\partial(v)f)g) = -Res_\Delta (f(\partial(v)g)).$$

If $o\sigma \in OB(\Delta)$ is an ordered basis, an important functional $Res^{o\sigma}$ can be defined on R_Δ : the **iterated residue** with respect to the ordered basis $o\sigma$. If we write an element $z \in V$ on the basis $o\sigma = (\alpha_1, \alpha_2, \dots, \alpha_r)$ as $z = (z_1, \dots, z_r)$, then

$$Res^{o\sigma}(f) = Res_{z_1=0}(Res_{z_2=0} \dots (Res_{z_r=0} f(z_1, z_2, \dots, z_r)) \dots).$$

The map $Res^{o\sigma}$ depends on the order $o\sigma$ chosen on σ and not only on the basis σ underlying $o\sigma$. The restriction of the functional $Res^{o\sigma}$ to S_Δ is called $r^{o\sigma}$. We have

$$(3) \quad Res^{o\sigma} = r^{o\sigma} Res_\Delta.$$

Indeed, we have only to check that $Res^{o\sigma}$ vanishes on derivatives. If $o\sigma = (\alpha_1, \alpha_2, \dots, \alpha_r)$ and $z = (z_1, \dots, z_r)$, the iterated residue $Res^{o\sigma}$ will vanish at the step $Res_{z_j=0}$ on $\frac{\partial}{\partial z_j} R_\Delta$.

Recall the following definition of A. Szenes ([3], Definition 3.3).

Definition 4 *A diagonal basis is a subset OB of $OB(\Delta)$ such that*

- 1) *The set of underlying (unordered) bases forms a basis B of $\mathcal{B}(\Delta)$.*
- 2) *The dual basis to the basis $(\phi_\sigma, o\sigma \in OB)$ is the set of linear forms $(r^{o\sigma}, o\sigma \in OB)$:*

$$r^{o\sigma}(\phi_\sigma) = \delta_\sigma^\tau.$$

In Proposition 3.4 of [3], it is proved that a total order on Δ gives rise to a diagonal basis (this is reproved in more detail in [1], Proposition 14.)

In the one dimensional case, $S_\Delta = \mathbb{C}z^{-1}$, and the space $G = \sum_{k \leq -1} \mathbb{C}z^k$ of negative Laurent series is the space obtained from the function $1/z$ by successive derivations. In the case of several variables, we can also characterize the space generated by simple fractions under differentiation.

Let κ be a sequence of (not necessarily distinct) elements of Δ . The sequence κ is called **generating** if the $\alpha \in \kappa$ generate the vector space V^* .

We denote by G_Δ the subspace of R_Δ spanned by the

$$\phi_\kappa := \frac{1}{\prod_{\alpha \in \kappa} \alpha}$$

where κ is a generating sequence. Finally, we denote by $S(V)$ the ring of differential operators on V , with constant coefficients. This ring acts on $S(V^*)$ and on R_Δ .

Proposition 5 ([1], Theorem 1) *The space G_Δ is the $S(V)$ -submodule of R_Δ generated by S_Δ .*

For example, if $\Delta = \{z_1, z_2, z_1 + z_2\}$, we have

$$\frac{1}{z_1 z_2 (z_1 + z_2)} = -\frac{\partial}{\partial z_1} \left(\frac{1}{z_1 z_2} \right) + \left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \right) \left(\frac{1}{z_1 (z_1 + z_2)} \right).$$

In particular, every element of G_Δ can be expressed as a linear combination of elements

$$\frac{1}{\prod_{\alpha \in \sigma} \alpha^{n_\alpha}}$$

where σ is a basis and the n_α are positive integers.

For example, the above equality is equivalent to

$$\frac{1}{z_1 z_2 (z_1 + z_2)} = \frac{1}{z_1^2 z_2} - \frac{1}{z_1^2 (z_1 + z_2)}.$$

The ring $S(V^*)$ operates by multiplication on R_Δ . It is also useful to consider the action of the ring $\mathcal{D}(V)$ of differential operators with polynomial coefficients, generated by $S(V)$ and $S(V^*)$. The following lemma is an obvious corollary of the description of G_Δ .

Lemma 6 *The space R_Δ is generated by G_Δ as an $S(V^*)$ -module. It is generated by S_Δ as a $\mathcal{D}(V)$ -module.*

Consider now the space \mathcal{O} of holomorphic functions on V defined in a neighborhood of 0. Let $\mathcal{O}_\Delta = \Delta^{-1}\mathcal{O}$ be the space of meromorphic functions in a neighborhood of 0, with denominators products of elements of Δ . The space \mathcal{O}_Δ is a module for the action of differential operators with constant coefficients. Via the Taylor series at the origin of elements of \mathcal{O} , the residue $\text{Res}_\Delta f(z)$ has still a meaning if $f(z) \in \mathcal{O}_\Delta$; indeed, $\text{Res}_\Delta f(z) = 0$ if $f \in R_\Delta$ is homogeneous of degree $\neq -r$.

If $y \in V$ is sufficiently near 0 and $f \in \mathcal{O}_\Delta$, the function

$$(\mathcal{T}(y)f)(z) := f(z - y)$$

is still an element of \mathcal{O}_Δ . If moreover y is regular, then $f(z - y)$ is defined for $z = 0$, and thus is an element of \mathcal{O} .

If $f \in R_\Delta$, we denote by $m(f)$ the operator of multiplication by f :

$$(m(f)\phi)(z) := f(z)\phi(z).$$

It operates on \mathcal{O}_Δ . Finally, we denote by C the operator

$$(Cf)(z) := f(-z)$$

on \mathcal{O}_Δ .

Theorem 7 (*Kernel theorem*). *Let $A : R_\Delta \rightarrow \mathcal{O}_\Delta$ be an operator commuting with the action of differential operators with constant coefficients. For $y \in V$ regular, sufficiently near 0 and for $f \in G_\Delta$, we have the formula*

$$(Af)(y) = \text{Tr}_{S_\Delta} (\text{Res}_\Delta m(f)CT(y)A\text{Res}_\Delta).$$

More explicitly, choose a basis B of $\mathcal{B}(\Delta)$ and let $(\phi^\sigma, \sigma \in B)$ be the basis of S_Δ^ dual to the basis $(\phi_\sigma, \sigma \in B)$ of S_Δ . Then we have the kernel formula:*

$$(Af)(y) = \sum_{\sigma \in B} \langle \phi^\sigma, \text{Res}_\Delta (f(z)A_\sigma(y-z)) \rangle$$

where $A_\sigma(z) = A(\phi_\sigma)(z)$.

Concretely, this formula means the following. Let f be homogeneous of degree d . We fix y regular and small. The function $z \mapsto A_\sigma(y-z)$ is defined near $z=0$. The Jeffrey-Kirwan residue Res_Δ of the function $z \mapsto f(z)A_\sigma(y-z)$ is a function of z belonging to the space S_Δ . We pair it with the linear form ϕ^σ on S_Δ and obtain a certain complex number depending on y . More precisely, consider the Taylor expansion

$$A_\sigma(y-z) = A_\sigma(y) + \sum_{j=1}^{\infty} A_\sigma^j(y, z)$$

where $A_\sigma^j(y, z)$ is the part of the Taylor expansion at 0 of the holomorphic function $z \mapsto A_\sigma(y-z)$, which is homogeneous of degree j in z . We have

$$A_\sigma^j(y, z) = (-1)^j \sum_{(k), |(k)|=j} A_\sigma^{(k)}(y) \frac{z^{(k)}}{(k)!}$$

where $(k) = (k_1, \dots, k_r)$ is a multi-index, and $A_\sigma^{(k)}(y) = ((\frac{\partial}{\partial y})^{(k)} A_\sigma)(y)$. Then, as the Jeffrey-Kirwan residue vanishes on homogeneous terms of degree not equal to $-r$, we obtain

$$\begin{aligned} \text{Res}_\Delta (f(z)A_\sigma(y-z)) &= \text{Res}_\Delta (f(z)A_\sigma^{-d-r}(y, z)) \\ &= (-1)^{d+r} \sum_{(k), |(k)|=-d-r} A_\sigma^{(k)}(y) \text{Res}_\Delta \left(f(z) \frac{z^{(k)}}{(k)!} \right). \end{aligned}$$

Thus, $\langle \phi^\sigma, \text{Res}_\Delta (f(z)A_\sigma(y-z)) \rangle$ is equal to

$$(-1)^{d+r} \sum_{(k), |(k)|=-d-r} A_\sigma^{(k)}(y) \langle \phi^\sigma, \text{Res}_\Delta \left(f(z) \frac{z^{(k)}}{(k)!} \right) \rangle.$$

Set $c_\sigma^{(k)}(f) = \langle \phi^\sigma, \text{Res}_\Delta \left(f(z) \frac{z^{(k)}}{(k)!} \right) \rangle$. Let $P_\sigma^f(\frac{\partial}{\partial y})$ be the differential operator with constant coefficients defined by

$$P_\sigma^f\left(\frac{\partial}{\partial y}\right) = (-1)^{d+r} \sum_{(k), |(k)|=-d-r} c_\sigma^{(k)}(f) \left(\frac{\partial}{\partial y}\right)^{(k)}.$$

Then P_σ^f depends linearly on f , and

$$\langle \phi^\sigma, \text{Res}_\Delta (f(z)A_\sigma(y-z)) \rangle = (P_\sigma^f\left(\frac{\partial}{\partial y}\right)A_\sigma)(y).$$

The claim of the theorem is that

$$(Af)(y) = \sum_{\sigma \in B} P_\sigma^f\left(\frac{\partial}{\partial y}\right) \cdot A_\sigma(y).$$

We now prove this theorem.

Proof. Define an operator $A' : R_\Delta \rightarrow \mathcal{O}_\Delta$ by

$$(A'f)(y) = \sum_{\sigma \in B} \langle \phi^\sigma, \text{Res}_\Delta (f(z)A_\sigma(y-z)) \rangle.$$

We first check that A' commutes with the action of differential operators with constant coefficients. Using the equation

$$(\partial_y(v)\phi)(y-z) = -(\partial_z(v)\phi)(y-z)$$

and the main property (2) of Res_Δ , we obtain

$$\begin{aligned}\partial_y(v) \cdot \langle \phi^\sigma, Res_\Delta(f(z)A_\sigma(y-z)) \rangle &= \langle \phi^\sigma, Res_\Delta(f(z)(\partial_y(v) \cdot A_\sigma(y-z))) \rangle \\ &= -\langle \phi^\sigma, Res_\Delta(f(z)(\partial_z(v) \cdot A_\sigma(y-z))) \rangle = \langle \phi^\sigma, Res_\Delta(\partial_z(v) \cdot f)A_\sigma(y-z) \rangle.\end{aligned}$$

It remains to see that A and A' coincide on S_Δ . For this, we will use the following formula. If P is a polynomial and ϕ a simple fraction, then

$$(4) \quad Res_\Delta(P\phi) = P(0)\phi.$$

To see this, recall that the function ϕ is homogeneous of degree $-r$. As $P \in S(V^*)$, $P - P(0)$ is a sum of homogeneous terms of positive degree. Thus, for homogeneity reasons, $Res_\Delta((P - P(0))\phi) = 0$.

Let y be regular and let $\sigma, \tau \in B$. As the function $z \rightarrow A_\sigma(y-z)$ is an element of \mathcal{O} , we obtain by formula (4),

$$Res_\Delta(\phi_\tau(z)A_\sigma(y-z)) = A_\sigma(y)\phi_\tau(z).$$

Thus

$$\begin{aligned}A'(\phi_\tau)(y) &= \sum_{\sigma \in B} \langle \phi^\sigma, Res_\Delta(\phi_\tau(z)A_\sigma(y-z)) \rangle \\ &= \sum_{\sigma \in B} \langle \phi^\sigma, \phi_\tau \rangle A_\sigma(y) = \sum_{\sigma \in B} \delta_\sigma^\tau A_\sigma(y) = A_\tau(y) = A(\phi_\tau)(y).\end{aligned}$$

Choosing a diagonal basis OB and using Equation 3, we obtain an iterated residue fomula for $(Af)(y)$:

Corollary 8 *For any diagonal basis OB of $\mathcal{B}(\Delta)$, we have for $f \in G_\Delta$:*

$$(Af)(y) = \sum_{o\sigma \in OB} Res^{o\sigma}(f(z)A_\sigma(y-z))$$

where $A_\sigma(z) = A(\phi_\sigma)(z)$.

Corollary 8 applies to the identity operator $A : R_\Delta \rightarrow R_\Delta$. If $f \in G_\Delta$, we obtain $f(y) = \sum_{o\sigma \in OB} Res^{o\sigma}(f(z)\phi_\sigma(y-z))$. But if $f \in NG_\Delta$ then clearly $Res^{o\sigma}(f(z)\phi_\sigma(y-z)) = 0$ as the Taylor series of $f(z)\phi_\sigma(y-z)$ at $z = 0$ is also in NG_Δ . As a consequence, we obtain a formula for the Jeffrey-Kirwan residue as a function of iterated residues:

Lemma 9 *For any $f \in R_\Delta$, we have*

$$(Res_\Delta f)(y) = \sum_{o\sigma \in OB} Res^{o\sigma}(f)\phi_\sigma(y).$$

Similarly, if $Z : R_\Delta \rightarrow \mathcal{O}$ is an operator commuting with the action of differential operators with constant coefficients, the formula

$$Z(f)(y) = Tr_{S_\Delta} (Res_\Delta m(f)CT(y)ZRes_\Delta)$$

is valid for **all** elements $y \in V$ sufficiently near 0 and for all $f \in G_\Delta$. In particular, we have the following

Proposition 10 *Let $Z : R_\Delta \rightarrow \mathcal{O}$ be an operator commuting with the action of differential operators with constant coefficients. Then we have, for $f \in G_\Delta$,*

$$Z(f)(0) = Tr_{S_\Delta} (Res_\Delta m(f)CZRes_\Delta)$$

where $(CZ)(\phi)(z) = Z(\phi)(-z)$.

Choosing a diagonal basis of $OB(\Delta)$, we can express the preceding formula as a residue formula in several variables:

$$Z(f)(0) = \sum_{o\sigma \in OB} Res^{o\sigma}(f(z)Z_\sigma(-z))$$

with $Z_\sigma(z) = Z(\phi_\sigma)(z)$.

For later use, we prove a vanishing property of the linear form $Res^{o\sigma}$. Let $o\sigma$ be an ordered basis. We write $o\sigma = (\alpha_1, \alpha_2, \dots, \alpha_r)$ and $z = (z_1, z_2, \dots, z_r)$. Set $o\sigma' = (\alpha_2, \dots, \alpha_r)$ and $z' = (z_2, \dots, z_r)$; then $z = (z_1, z')$. Let $\psi(z')$ in $\mathcal{O}_{\Delta'}$ be a meromorphic function with denominator a product of linear forms $\alpha(z')$ where $\alpha \in \Delta$ is not a multiple of α_1 .

Lemma 11 *For any $f \in G_\Delta$, and any $\psi \in \mathcal{O}_{\Delta'}$,*

$$Res^{o\sigma} \left(\frac{1}{z_1} f(z_1, z') \psi(z') \right) = 0.$$

Proof. We have

$$Res^{o\sigma} \left(\frac{1}{z_1} f(z_1, z') \psi(z') \right) = Res_{z_1=0} \left(\frac{1}{z_1} Res^{o\sigma'} (f(z_1, z') \psi(z')) \right).$$

In computing $Res^{o\sigma'} (f(z_1, z') \psi(z'))$, the variable z_1 is fixed to a non-zero value. The result $Res^{o\sigma'} (f(z_1, z') \psi(z'))$ is a meromorphic function of z_1 . It is thus sufficient to prove that $Res^{o\sigma'} (f(z_1, z') \psi(z'))$ belongs to the space $G = \sum_{k \leq -1} \mathbb{C} z_1^k$.

We check this for $f = \phi_\kappa$ where

$$\phi_\kappa(z) = \frac{1}{\prod_{\alpha \in \kappa} \langle \alpha, z \rangle}$$

and κ is a generating sequence. Let

$$\kappa_1 := \{\alpha \in \kappa, \langle \alpha, (z_1, 0) \rangle \neq 0\}$$

and

$$\kappa' = \{\alpha \in \kappa, \langle \alpha, (z_1, 0) \rangle = 0\}.$$

As κ is generating, the set κ_1 is not empty. We fix $z_1 \neq 0$. We have

$$\phi_\kappa(z_1, z') \psi(z') = \phi_{\kappa_1}(z_1, z') \phi_{\kappa'}(z') \psi(z')$$

and $\phi_{\kappa'} \in \mathcal{O}_{\Delta'}$. For $\alpha \in \kappa_1$, we set $\langle \alpha, (z_1, z') \rangle = c_\alpha z_1 + \langle \beta, z' \rangle$, with $c_\alpha \neq 0$. We consider the Taylor expansion at $z' = 0$ of the holomorphic function of z' :

$$\frac{1}{\langle \alpha, (z_1, z') \rangle} = \frac{1}{c_\alpha z_1 + \langle \beta, z' \rangle} = \frac{1}{c_\alpha z_1 (1 + \frac{\langle \beta, z' \rangle}{c_\alpha z_1})}.$$

This is of the form

$$\sum_{k=1}^{\infty} z_1^{-k} P_{k-1}(z')$$

where $P_{k-1}(z')$ is homogeneous of degree $k-1$ in z' . Let $n = |\kappa_1|$, then $n \geq 1$. We see that the function

$$z' \mapsto \phi_{\kappa_1}(z_1, z') = \frac{1}{\prod_{\alpha \in \kappa_1} \langle \alpha, (z_1, z') \rangle}$$

has a Taylor expansion of the form

$$\sum_{k \geq n} z_1^{-k} Q_{k-1}(z')$$

where $Q_{k-1}(z')$ is homogeneous of degree $k-1$ in z' . Thus

$$Res^{o\sigma'}(\phi_{\kappa_1}(z_1, z')\phi_{\kappa'}(z')\psi(z')) = \sum_{k \geq n} z_1^{-k} Res^{o\sigma'}(Q_{k-1}(z')\phi_{\kappa'}(z')\psi(z')).$$

Via the Taylor series at $z' = 0$, the function $\phi_{\kappa'}(z')\psi(z')$ can be expressed as an infinite sum of homogeneous elements with finitely many negative degrees. As the iterated residue $Res^{o\sigma'}$ vanishes on elements of degree not equal to $-(r-1)$, and $Q_{k-1}(z')$ is homogeneous of degree $k-1$, we see that the sum is finite and that $Res^{o\sigma'}(\phi_{\kappa_1}(z_1, z')\phi_{\kappa'}(z')\psi(z'))$ is in the space G as claimed.

3 Eisenstein series

Results of the second section will be used for a complex vector space which is the complexification of a real vector space. Thus we slightly change the notation in this section.

Let V be a **real** vector space of dimension r equipped with a lattice N . The complex vector space $V_{\mathbb{C}}$ is the space to which we will apply the results of Section 2.

We consider the dual lattice $M = N^*$ to N . We consider the compact torus $T = iV/(2i\pi N)$ and its complexification $T_{\mathbb{C}} = V_{\mathbb{C}}/(2i\pi N)$. The projection map $V_{\mathbb{C}} \rightarrow T_{\mathbb{C}}$ is denoted by the exponential notation $v \rightarrow e^v$. If $\{e^1, e^2, \dots, e^r\}$ is a \mathbb{Z} -basis of N , we write an element of $V_{\mathbb{C}}$ as $z = z_1 e^1 + z_2 e^2 + \dots + z_r e^r$ with $z_j \in \mathbb{C}$. We can identify $T_{\mathbb{C}}$ with $\mathbb{C}^* \times \mathbb{C}^* \times \dots \times \mathbb{C}^*$ by $z \mapsto (e^{z_1}, e^{z_2}, \dots, e^{z_r})$.

If $m \in M$, we denote by e^m the character of T defined by $\langle e^m, e^v \rangle = e^{\langle m, v \rangle}$. We extend e^m to a holomorphic character of the complex torus $T_{\mathbb{C}}$. The ring of holomorphic functions on $T_{\mathbb{C}}$ generated by the functions e^m is denoted by $R(T)$. A quotient of two elements of $R(T)$ is called a **rational function** on the complex torus $T_{\mathbb{C}}$. Via the exponential map $V_{\mathbb{C}} \rightarrow T_{\mathbb{C}}$, a function on $T_{\mathbb{C}}$ will be sometimes identified with a function on $V_{\mathbb{C}}$, invariant under translation by the lattice $2i\pi N$. If $\{e^1, e^2, \dots, e^r\}$ is a \mathbb{Z} -basis of N , a rational function on $T_{\mathbb{C}}$ written in exponential coordinates is a rational function of $e^{z_1}, e^{z_2}, \dots, e^{z_r}$. We say briefly that it is a rational function of e^z .

Let us consider a finite set Δ of non-trivial characters of T . We identify Δ with a subset of M ; for $\alpha \in \Delta$, we denote by e^α the corresponding character of $T_{\mathbb{C}}$.

Definition 12 We denote by $R(T)_\Delta$ the subring of rational functions on T generated by $R(T)$ and the inverses of the functions $1 - e^{-\alpha}$ with $\alpha \in \Delta$.

Observe that R_Δ is left unchanged when each element of Δ is replaced by a non-zero scalar multiple, but that $R(T)_\Delta$ strictly increases when (say) each $\alpha \in \Delta$ is replaced by 2α . We assume from now on that all elements of Δ are indivisible in the lattice M .

Via the exponential map, we consider elements of $R(T)_\Delta$ as periodic meromorphic functions on $V_\mathbb{C}$. On $V_\mathbb{C}$, the function

$$\frac{\langle \alpha, z \rangle}{1 - e^{-\langle \alpha, z \rangle}}$$

is defined at $z = 0$, so is an element of \mathcal{O} . Writing

$$\frac{1}{1 - e^{-\langle \alpha, z \rangle}} = \frac{1}{\langle \alpha, z \rangle} \frac{\langle \alpha, z \rangle}{1 - e^{-\langle \alpha, z \rangle}},$$

we see that $R(T)_\Delta$ is contained in \mathcal{O}_Δ . We see furthermore from the formula

$$\frac{d}{dz} \frac{1}{1 - e^{-z}} = \frac{1}{(1 - e^z)(1 - e^{-z})} = \frac{-e^{-z}}{(1 - e^{-z})^2}$$

that $R(T)_\Delta \subset \mathcal{O}_\Delta$ is stable under differentiation.

Our aim is to find a natural map from R_Δ to $R(T)_\Delta$ commuting with the action of differential operators with constant coefficients. In particular, we want to force a rational function of $z \in V_\mathbb{C}$ to become periodic, so that it is natural to define Eisenstein series

$$E(f)(z) = \sum_{n \in N} f(z + 2i\pi n).$$

We need to be more careful, as the sum is usually not convergent for an arbitrary $f \in R_\Delta$. We will introduce an oscillating factor $e^{\langle t, 2i\pi n \rangle}$ with $t \in V^*$ in front of each term of this infinite sum.

Let

$$U_\Delta = \{z \in V_\mathbb{C}, \langle \alpha, z + 2i\pi n \rangle \neq 0 \text{ for all } n \in N \text{ and for all } \alpha \in \Delta\}.$$

Then $R(T)_\Delta$ consists of periodic holomorphic functions on U_Δ .

Let $f \in R_\Delta$, then $f(z + 2i\pi n)$ is defined for each $n \in N$ if $z \in U_\Delta$. For $z \in U_\Delta$, we consider the function on V^* defined by

$$t \mapsto \sum_{n \in N} e^{\langle t, z + 2i\pi n \rangle} f(z + 2i\pi n).$$

If $n \mapsto f(z + 2i\pi n)$ is sufficiently decreasing at infinity, the series will be absolutely convergent and sum up to a continuous function of t with value at $t = 0$ equal to

$$\sum_{n \in N} f(z + 2i\pi n).$$

In any case, it is easy to see that this series of functions of t converges to a generalized function of t :

Proposition 13 *For each $f \in R_\Delta$ and $z \in U_\Delta$, the function on V^* defined by*

$$t \mapsto \sum_{n \in N} e^{\langle t, z + 2i\pi n \rangle} f(z + 2i\pi n)$$

is well defined as a generalized function of t , which depends holomorphically on z for z in the open set U_Δ .

Proof. Indeed, if $s(t)$ is a smooth function on V^* with compact support, consider the series

$$\sum_{n \in N} f(z + 2i\pi n) \int_{V^*} e^{\langle t, z + 2i\pi n \rangle} s(t) dt = \sum_{n \in N} c(z, n) f(z + 2i\pi n).$$

The coefficient

$$c(z, n) = \int_{V^*} e^{2i\pi \langle t, n \rangle} e^{\langle t, z \rangle} s(t) dt$$

is rapidly decreasing in n , as the function $t \mapsto e^{\langle t, z \rangle} s(t)$ is smooth and compactly supported. Thus $c(z, n) f(z + 2i\pi n)$ is also a rapidly decreasing function of n . Furthermore $c(z, n) f(z + 2i\pi n)$ depends holomorphically on $z \in U_\Delta$. So the result of the summation

$$\sum_{n \in N} c(z, n) f(z + 2i\pi n)$$

exists and is a holomorphic function of z .

We write

$$E(f)(t, z) = \sum_{n \in N} e^{\langle t, z + 2i\pi n \rangle} f(z + 2i\pi n)$$

for this generalized function of t depending holomorphically on z . We will analyze this function of (t, z) , $t \in V^*$, $z \in U_\Delta$.

We summarize first some of the obvious properties of $E(f)(t, z)$.

Proposition 14 *The following equations are satisfied:*

1) *For every $P \in S(V^*)$ and $f \in R_\Delta$,*

$$E(Pf)(t, z) = P(\partial_t)E(f)(t, z).$$

2) *For every $v \in V$ and $f \in R_\Delta$,*

$$E(\partial(v)f)(t, z) = \partial_z(v)E(f)(t, z) - \langle t, v \rangle E(f)(t, z).$$

3) *For every $m \in M$ and $z \in U_\Delta$,*

$$E(f)(t + m, z) = e^{\langle m, z \rangle} E(f)(t, z).$$

As R_Δ is generated by S_Δ under the action of $S(V)$ and $S(V^*)$, we see that the operator E is completely determined by the functions $E(\phi_\sigma)(t, z)$ ($\sigma \in \mathcal{B}(\Delta)$).

A **wall** of Δ is a hyperplane of V^* generated by $r - 1$ linearly independent vectors of Δ . We consider the system of affine hyperplanes generated by the walls of Δ together with their translates by M (the dual lattice of N). We denote by $V_{\Delta, \text{areg}}^*$ the complement of the union of these affine hyperplanes. A connected component of $V_{\Delta, \text{areg}}^*$ will be called an **alcove** and will be denoted by \mathfrak{a} .

Proposition 15 *The function $E(f)(t, z)$ is smooth when t varies on $V_{\Delta, \text{areg}}^*$, and $z \in U_\Delta$. More precisely, let \mathfrak{a} be an alcove. Assume that f is homogeneous of degree d . Then, on the open set $\mathfrak{a} \times U_\Delta$, the function $E(f)(t, z)$ is a polynomial in t of degree at most $-d - r$, with coefficients in $R(T)_\Delta$.*

Proof. Consider first the one variable case. The set $V_{\Delta, \text{areg}}^*$ is $\mathbb{R} - \mathbb{Z}$. Let $[t]$ be the integral part of t . Fix $z \in \mathbb{C} - 2i\pi\mathbb{Z}$. Consider the locally constant function of $t \in \mathbb{R} - \mathbb{Z}$ defined by

$$t \mapsto \frac{e^{[t]z}}{1 - e^{-z}}.$$

We extend this function as a locally L^1 -function on \mathbb{R} (defined except on the set \mathbb{Z} of measure 0).

Lemma 16 *We have the equality of generalized functions of t :*

$$\sum_{n \in \mathbb{Z}} \frac{e^{t(z+2i\pi n)}}{z+2i\pi n} = \frac{e^{[t]z}}{1-e^{-z}}.$$

Proof. We compute the derivative in t of the left hand side. It is equal to

$$\sum_{n \in \mathbb{Z}} e^{t(z+2i\pi n)} = e^{tz} \delta_{\mathbb{Z}}(t)$$

where $\delta_{\mathbb{Z}}$ is the delta function of the set of integers.

We compute the derivative in t of the right hand side. This function of t is constant on each interval $(n, n+1)$. The jump at the integer n is

$$\frac{e^{nz}}{1-e^{-z}} - \frac{e^{(n-1)z}}{1-e^{-z}} = e^{nz}.$$

It follows that the derivative in t of the right hand side is also equal to $e^{tz} \delta_{\mathbb{Z}}(t)$. Thus

$$\sum_{n \in \mathbb{Z}} \frac{e^{t(z+2i\pi n)}}{z+2i\pi n} = c(z) + \frac{e^{[t]z}}{1-e^{-z}}$$

where $c(z)$ is a constant. We verify that $c(z)$ is equal to 0 by using periodicity properties in t . It is clear that

$$e^{-tz} \sum_{n \in \mathbb{Z}} \frac{e^{t(z+2i\pi n)}}{z+2i\pi n} = \sum_{n \in \mathbb{Z}} \frac{e^{2i\pi nt}}{z+2i\pi n}$$

is a periodic function of t as is

$$e^{-tz} \frac{e^{[t]z}}{1-e^{-z}} = \frac{e^{([t]-t)z}}{1-e^{-z}}.$$

It follows that $e^{-tz} c(z)$ is also a periodic function of t . This implies $c(z) = 0$.

Consider now, for $k \in \mathbb{Z}$,

$$E_k(t, z) = \sum_{n \in \mathbb{Z}} e^{t(z+2i\pi n)} (z+2i\pi n)^k.$$

We just saw that

$$E_{-1}(t, z) = \frac{e^{[t]z}}{1-e^{-z}}.$$

To determine $E_k(t, z)$ for $k \leq -1$, we use the differential equation in z

$$\partial_z E_k(t, z) = tE_k(t, z) + kE_{k-1}(t, z).$$

Using decreasing induction over k , we see that $E_k(t, z)$ is a L^1 -function of t , equal to a polynomial function of t of degree $-k - 1$ on each interval $(n, n + 1)$ and with coefficients rational functions of e^z . For example, we obtain the value of the convergent series

$$\sum_n \frac{e^{t(z+2i\pi n)}}{(z+2i\pi n)^2} = (t - [t]) \frac{e^{[t]z}}{1 - e^{-z}} - \frac{e^{[t]z}}{(1 - e^{-z})(1 - e^z)}.$$

When $k \geq 0$, we use the differential equation

$$\partial_t E_k(t, z) = E_{k+1}(t, z)$$

so that, as we have already used,

$$E_0(t, z) = \sum_{n \in \mathbb{Z}} e^{t(z+2i\pi n)} = e^{tz} \delta_{\mathbb{Z}}(t).$$

More generally, $E_k(t, z) = (\partial_t)^k(e^{tz} \delta_{\mathbb{Z}}(t))$ is supported on \mathbb{Z} , in particular is identically 0 on $\mathbb{R} - \mathbb{Z}$.

We return to the proof of Proposition 15. For a simple fraction ϕ , consider the function

$$t \mapsto E(\phi)(t, z).$$

We first prove that it is a locally L^1 -function, which is constant when t varies in an alcove.

Let $\sigma = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be a basis of Δ . Let $t \in V^*$. If $t = \sum_j t_j \alpha_j$ is the decomposition of t on the basis σ , set $[t]_\sigma = \sum_j [t_j] \alpha_j$. The function $t \mapsto [t]_\sigma$ is constant when t varies in an alcove. Consider the sublattice

$$M_\sigma = \bigoplus_{\alpha \in \sigma} \mathbb{Z}\alpha \subseteq M.$$

We say that σ is a \mathbb{Z} -basis, if $M_\sigma = M$. In general, the quotient M/M_σ is a finite set; let \mathcal{R} be a set of representatives of this quotient. We can choose \mathcal{R} in the following standard way. We consider the box

$$Q_\sigma = \bigoplus_{\alpha \in \sigma} [0, 1[\alpha = \{u \in V^*, [u]_\sigma = 0\}.$$

Then we can take

$$\mathcal{R} = Q_\sigma \cap M = \{u \in M, [u]_\sigma = 0\}.$$

Define

$$\mathcal{R}(t, \sigma) = (t - Q_\sigma) \cap M = \{u \in M, [t - u]_\sigma = 0\}.$$

The set $\mathcal{R}(t, \sigma)$ is also a set of representatives of M/M_σ . If σ is a \mathbb{Z} -basis of M , this set is reduced to the single element $[t]_\sigma$. Remark that the set $\mathcal{R}(t, \sigma)$ is constant when t varies in an alcove \mathfrak{a} . We denote it by $\mathcal{R}(\mathfrak{a}, \sigma)$.

Definition 17 *If \mathfrak{a} is an alcove, and σ a basis of Δ , we set*

$$F_\sigma^\mathfrak{a} = |M/M_\sigma|^{-1} \frac{\sum_{m \in \mathcal{R}(\mathfrak{a}, \sigma)} e^m}{\prod_{\alpha \in \sigma} (1 - e^{-\alpha})}.$$

Thus an alcove \mathfrak{a} together with a basis $\sigma \in \mathcal{B}(\Delta)$ produces a particular element $F_\sigma^\mathfrak{a}$ of $R(T)_\Delta$.

Consider on the set $V_{\Delta, \text{areg}}^*$ the locally constant function of t defined by $F_\sigma(t, z) = F_\sigma^\mathfrak{a}(z)$ when t is in the alcove \mathfrak{a} . This defines a locally L^1 -function of t , still denoted by $F_\sigma(t, z)$, defined except on the set $V^* - V_{\Delta, \text{areg}}^*$ of measure 0. This locally L^1 -function of t defines a generalized function of t which depends holomorphically on z .

Lemma 18 *We have the equality of generalized functions of $t \in V^*$:*

$$E(\phi_\sigma)(t, z) = F_\sigma(t, z).$$

Proof. If σ is a \mathbb{Z} -basis of M , this follows from the formula in dimension 1. In general, we consider $M_\sigma \subseteq M$ and the dual lattice $N_\sigma = M_\sigma^*$. Then $N \subseteq N_\sigma$. We set

$$E_\sigma(\phi_\sigma)(t, z) := \sum_{\ell \in N_\sigma} e^{\langle t, z + 2i\pi\ell \rangle} \phi_\sigma(z + 2i\pi\ell).$$

For any set of representatives \mathcal{R} of M/M_σ , we have: $\sum_{u \in \mathcal{R}} e^{-\langle u, 2i\pi\ell \rangle} = 0$ if

$\ell \in N_\sigma$ is not in N , while this sum equal $|M/M_\sigma|$ if $n \in N$. Thus,

$$\begin{aligned}
E(\phi_\sigma)(t, z) &= \sum_{n \in N} \phi_\sigma(z + 2i\pi n) e^{\langle t, z + 2i\pi n \rangle} \\
&= \sum_{\ell \in N_\sigma} \phi_\sigma(z + 2i\pi \ell) e^{\langle t, z + 2i\pi \ell \rangle} (|M/M_\sigma|^{-1} \sum_{u \in \mathcal{R}} e^{-\langle u, 2i\pi \ell \rangle}) \\
&= |M/M_\sigma|^{-1} \sum_{u \in \mathcal{R}} \sum_{\ell \in N_\sigma} \phi_\sigma(z + 2i\pi \ell) e^{\langle t-u, z + 2i\pi \ell \rangle} e^{\langle u, z \rangle} \\
&= |M/M_\sigma|^{-1} \sum_{u \in \mathcal{R}} e^{\langle u, z \rangle} E_\sigma(\phi_\sigma)(t - u, z).
\end{aligned}$$

This holds as an equality of generalized functions of t . Further, we have by the one-dimensional case:

$$E_\sigma(\phi_\sigma)(t, z) = \frac{e^{\langle [t]_\sigma, z \rangle}}{\prod_{\alpha \in \sigma} (1 - e^{-\langle \alpha, z \rangle})}.$$

It follows that $E(\phi_\sigma)(t, z)$ is a locally L^1 -function of t , as is $E_\sigma(\phi_\sigma)$. It remains to determine the value of this function when t is in an alcove. For $m \in M_\sigma$, we have

$$E_\sigma(\phi_\sigma)(t + m, z) = e^{\langle m, z \rangle} E_\sigma(\phi_\sigma)(t, z),$$

so that the sum $\sum_{u \in \mathcal{R}} e^{\langle u, z \rangle} E_\sigma(\phi_\sigma)(t - u, z)$ is independent of the choice of the system of representatives \mathcal{R} of M/M_σ . We choose $\mathcal{R} = \mathcal{R}(t, \sigma)$. Then

$$E(\phi_\sigma)(t, z) = |M/M_\sigma|^{-1} \frac{\sum_{u \in \mathcal{R}(t, \sigma)} e^{\langle u, z \rangle}}{\prod_{\alpha \in \sigma} (1 - e^{-\langle \alpha, z \rangle})}$$

because $[t - u]_\sigma = 0$ for all $u \in \mathcal{R}(t, \sigma)$.

Every function $f \in R_\Delta$, homogeneous of degree d , is obtained from an element of S_Δ by the action of a differential operator with polynomial coefficients. This operator is of degree $d + r$, if multiplication by z_j is given degree 1, while derivation $\frac{\partial}{\partial z_j}$ is given degree -1 . Using Proposition 14, we see that Proposition 15 follows from the fact that the function $t \mapsto E(\phi_\sigma)(t, z)$ is constant on each alcove.

From Proposition 15, we see that there exist functions $\phi_{(k)}^{\mathfrak{a}}(z) \in R(T)_\Delta$ such that we have the equality for t in the alcove \mathfrak{a} :

$$E(f)(t, z) = \sum_{n \in N} e^{\langle t, z + 2i\pi n \rangle} f(z + 2i\pi n) = \sum_{(k)} t^{(k)} \phi_{(k)}^{\mathfrak{a}}(z)$$

where the sum is over a finite number of multi-indices (k) . This defines an operator

$$E^t : R_\Delta \rightarrow R(T)_\Delta, f \mapsto E(f)(t, z)$$

obtained by fixing the regular value t .

The operator E^t satisfies the following relation, which is just the relation (2) in Proposition 14: For $v \in V$ and $f \in R_\Delta$,

$$E^t(\partial(v)f)(z) = \partial_z(v)E^t(f)(z) - \langle t, v \rangle E^t(f)(z).$$

Let B be a basis of $\mathcal{B}(\Delta)$. Let $(\phi_\sigma, \sigma \in B)$ be the corresponding basis of S_Δ and $(\phi^\sigma, \sigma \in B)$ the dual basis of S_Δ^* . For $\sigma \in B$, and an alcove \mathfrak{a} , consider the element $F_\sigma^\mathfrak{a}$ of $R(T)_\Delta \subset \mathcal{O}_\Delta$ associated to σ, \mathfrak{a} . We obtain a kernel formula for the operator E^t :

Theorem 19 *Let $f \in G_\Delta$. For $y \in U_\Delta$ and $t \in \mathfrak{a}$, we have:*

$$\begin{aligned} E^t(f)(y) &= \text{Tr}_{S_\Delta} \left(\text{Res}_\Delta m(e^{\langle t, \cdot \rangle} f) C\mathcal{T}(y) E^t \text{Res}_\Delta \right) \\ &= \sum_{\sigma \in B} \langle \phi^\sigma, \text{Res}_\Delta (e^{\langle t, z \rangle} f(z) F_\sigma^\mathfrak{a}(y - z)) \rangle \end{aligned}$$

where $F_\sigma^\mathfrak{a}$ is given by Definition 17. If moreover B is the underlying basis of a diagonal basis OB , then

$$E^t(f)(y) = \sum_{o\sigma \in OB} \text{Res}^{o\sigma} (e^{\langle t, z \rangle} f(z) F_\sigma^\mathfrak{a}(y - z)).$$

Proof. By a method entirely similar to the proof of Theorem 1, we see that the operator

$$A^t(f)(y) = \sum_{\sigma \in B} \langle \phi^\sigma, \text{Res}_\Delta (e^{\langle t, z \rangle} f(z) F_\sigma^\mathfrak{a}(y - z)) \rangle$$

satisfies the relation

$$A^t(\partial(v)f)(z) = \partial_z(v)A^t(f)(z) - \langle t, v \rangle A^t(f)(z)$$

for $v \in V$, $f \in R_\Delta$. Thus to prove that $E^t = A^t$ on G_Δ , it is sufficient to prove that they coincide for $f = \phi_\tau$. In this case, we obtain

$$A^t(\phi_\tau)(y) = \sum_{\sigma \in B} \langle \phi^\sigma, \phi_\tau(z) \rangle F_\sigma^\mathfrak{a}(y) = F_\tau^\mathfrak{a}(y) = E^t(\phi_\tau)(y).$$

In view of the kernel formula for the Eisenstein series E^t , it is natural to introduce the following definition.

Definition 20 *The constant term of the Eisenstein series E^t is the linear form $f \rightarrow CT(f)(t)$ defined for $f \in R_\Delta$ and t in the alcove \mathfrak{a} by*

$$CT(f)(t) = Tr_{S_\Delta} (Res_\Delta m(e^{\langle t, \cdot \rangle} f) CE^t Res_\Delta).$$

More explicitly, if OB is a diagonal basis of $\mathcal{B}(\Delta)$, then

$$CT(f)(t) = \sum_{o\sigma \in OB} Res^{o\sigma} (e^{\langle t, z \rangle} f(z) F_\sigma^{\mathfrak{a}}(-z)).$$

4 Partial Eisenstein series

Let $N_{reg} = N \cap V_{reg}$ be the set of regular elements of N . The aim of this section is to prove that the function

$$E_{N_{reg}}(f)(t, z) = \sum_{n \in N_{reg}} e^{\langle t, z + 2i\pi n \rangle} f(z + 2i\pi n)$$

is analytic in (t, z) when t is in an alcove and $z \in V_{\mathbb{C}}$ is close to 0. In the next section we will prove Szenes residue formula for

$$E_{N_{reg}}(f)(t, 0) = \sum_{n \in N_{reg}} e^{\langle t, 2i\pi n \rangle} f(2i\pi n).$$

Let Γ be a subset of N . We can define, for $f \in R_\Delta$, the generalized function of t

$$E_\Gamma(f)(t, z) = \sum_{n \in \Gamma} e^{\langle t, z + 2i\pi n \rangle} f(z + 2i\pi n).$$

Introduce the set

$$U_{\Delta, \Gamma} = \{z \in V_{\mathbb{C}}, \langle \alpha, z + 2i\pi n \rangle \neq 0 \text{ for all } \alpha \in \Delta \text{ and } n \in \Gamma\}.$$

The generalized function $E_\Gamma(f)(t, z)$ depends holomorphically on z , when $z \in U_{\Delta, \Gamma}$.

Let W be a rational subspace of V . Then $N \cap W$ is a lattice in W . Consider, for $f \in R_\Delta$,

$$E_{N \cap W}(f)(t, z) = \sum_{n \in N \cap W} e^{\langle t, z + 2i\pi n \rangle} f(z + 2i\pi n).$$

We analyze the singularities in (t, z) of $E_{N \cap W}(f)(t, z)$. If W is zero, then $E_{\{0\}}(f)(t, z) = e^{\langle t, z \rangle} f(z)$ is analytic in (t, z) when z is regular in $V_{\mathbb{C}}$. Assume that W is non-zero and consider the subspace W^{\perp} of V^* . Remark that, if $u \in M + W^{\perp}$, we have the relation

$$E_{N \cap W}(f)(t + u, z) = e^{\langle u, z \rangle} E_{N \cap W}(f)(t, z).$$

It is clear that the singular set of $E_{N \cap W}(f)(t, z)$ is stable by translation by $M + W^{\perp}$. Define a (W, Δ) -**wall** in V^* as an hyperplane generated by W^{\perp} together with $\dim W - 1$ vectors of Δ . We introduce the set $\mathcal{H}_{W, \Delta, M}^*$ consisting of the union of all (W, Δ) -walls and of their translates by elements of M . We define $V_{W, \Delta, a\text{reg}}^*$ as the complement of $\mathcal{H}_{W, \Delta, M}^*$ in V^* . This set $V_{W, \Delta, a\text{reg}}^*$ is invariant by translation by $M + W^{\perp}$.

Lemma 21 *For $f \in R_{\Delta}$, the function $E_{N \cap W}(f)(t, z)$ is analytic in (t, z) when t varies on $V_{W, \Delta, a\text{reg}}^*$ and $z \in U_{\Delta, N \cap W}$. Furthermore, if $t \in V_{W, \Delta, a\text{reg}}^*$ and z is near 0, the function $z \mapsto E_{N \cap W}(f)(t, z)$ defines an element of \mathcal{O}_{Δ} .*

Proof. Let σ be a basis of Δ . Although we are not able to give a nice formula for the function $E_{N \cap W}(\phi_{\sigma})(t, z)$, we can still obtain an inductive expression that suffices to give some informations on it. Consider the set $V_{W, \sigma, a\text{reg}}^*$, that is, the complement of the union of (W, σ) -walls together with their translates by M . Let $U_{\sigma, N \cap W}$ be the set of all $z \in V_{\mathbb{C}}$ such that $\langle \alpha, z + 2i\pi n \rangle \neq 0$ for all $\alpha \in \sigma$ and $n \in N \cap W$. The intersection of this set with a small neighborhood of 0 is contained in the complement of the union of the complex hyperplanes $\{z \in V_{\mathbb{C}}, \langle \alpha, z \rangle = 0\}$, for $\alpha \in \sigma$.

Lemma 22 *The function $E_{N \cap W}(\phi_{\sigma})(t, z)$ is analytic in $t \in V_{W, \sigma, a\text{reg}}^*$ and $z \in U_{\sigma, N \cap W}$. Furthermore, when $t \in V_{W, \sigma, a\text{reg}}^*$, the function*

$$z \mapsto \left(\prod_{\alpha \in \sigma} \langle \alpha, z \rangle \right) E_{N \cap W}(\phi_{\sigma})(t, z)$$

is holomorphic at $z = 0$.

We prove this by induction on the codimension of W . If $W = V$, this follows from the explicit formula for $E(\phi_{\sigma})(t, z)$. Let α be an indivisible element of M such that W is contained in the real hyperplane

$$H_{\alpha} = \{y \in V, \langle \alpha, y \rangle = 0\}.$$

We assume first that α is an element of σ . We number it the first vector α_1 of the basis σ . We set $\sigma' = (\alpha_2, \dots, \alpha_r)$, $z' = (z_2, \dots, z_r)$, etc; then $z = (z_1, z')$. Our subspace W is contained in $V' = V \cap \{z_1 = 0\}$. Thus, we have

$$E_{N \cap W}(\phi_\sigma)(t, z) = \sum_{n \in N \cap W} e^{\langle t, z + 2i\pi n \rangle} \phi_\sigma(z + 2i\pi n) = \frac{e^{t_1 z_1}}{z_1} E_{N' \cap W}(\phi_{\sigma'})(t', z').$$

By induction, $E_{N' \cap W}(\phi_{\sigma'})(t', z')$ is analytic in (t', z') for $z' \in U_{\sigma', N'}$, except if there exist $m' \in M'$ such that $t' + m'$ is in a hyperplane generated by $W^{\perp'}$ (the orthogonal of W in V') and some vectors of σ' . As $W^\perp = W^{\perp'} \oplus \mathbb{R}\alpha_1$, we see that the singular set of $E_{N \cap W}(\phi_\sigma)(t, z)$ is contained in $\mathcal{H}_{W, \sigma, M}^*$. Furthermore, the function

$$z_1 z_2 \cdots z_r E_{N \cap W}(\phi_\sigma)(t, z) = e^{t_1 z_1} z_2 \cdots z_r E_{N' \cap W}(\phi_{\sigma'})(t', z')$$

is holomorphic in z near $z = 0$.

Assume now that α is not an element of σ . We add it to the system Δ if α is not an element of Δ . Writing $\alpha = \sum_j c_j \alpha_j$, we obtain one of the Orlik-Solomon relations of the system $\Delta \cup \{\alpha\}$

$$\phi_\sigma = \sum_j c_j \phi_{\sigma^j}$$

where $\sigma^j = \sigma \cup \{\alpha\} - \{\alpha_j\}$. A (W, σ^j) -wall is a hyperplane of V^* generated by W^\perp and $\dim W - 1$ vectors of σ^j ; then these vectors are distinct from α , because $\alpha \in W^\perp$. Thus, all W -walls for the basis σ^j are also W -walls for the basis σ . By our first calculation, it follows that $E_{N \cap W}(\phi_{\sigma^j})(t, z)$ is analytic when t is not on a translate of a (W, σ) -wall. Moreover, we have

$$E_{N \cap W}(\phi_\sigma)(t, z) = \sum_j c_j E_{N \cap W}(\phi_{\sigma^j})(t, z),$$

so that the function

$$z \mapsto \langle \alpha, z \rangle \left(\prod_{j=1}^r \langle \alpha_j, z \rangle \right) E_{N \cap W}(\phi_\sigma)(t, z)$$

is holomorphic in z in a neighborhood of 0.

By the induction hypothesis applied to $W \subseteq V' = \{\alpha = 0\}$, the function $z \mapsto E_{N \cap W}(\phi_\sigma)(t, z)$ is holomorphic on a non-empty open subset of $V'_\mathbb{C}$. So

this function, considered as a function of $z \in V_{\mathbb{C}}$, has no pole along $\alpha = 0$. This proves Lemma 22, and hence Lemma 21 when f is a simple fraction. The operator $E_{N \cap W}$ satisfies also the commutation relation of Proposition 14. Thus, using differential operators with polynomial coefficients, we obtain the statement of Lemma 21 when f is any element in R_{Δ} .

Let I be a subset of Δ and let $W_I = \cap_{i \in I} H_{\alpha_i}$. This is a rational subspace of V , and the (W_I, Δ) -walls are some of the walls of Δ . Then it follows from Lemma 21 that $E_{N \cap W_I}(f)(t, z)$ is a fortiori analytic when $t \in V_{\text{areg}}^*$ and $z \in U_{\Delta}$.

Definition 23 *A subset Γ of N is **admissible**, if the characteristic function of Γ is a linear combination of characteristic functions of sets $N \cap W_I$, where I ranges over subsets of Δ .*

Then we have by Lemma 21:

Lemma 24 *If Γ is an admissible subset of N , the function $(t, z) \mapsto E_{\Gamma}(f)(t, z)$ is analytic when $t \in V_{\Delta, \text{areg}}^*$ and $z \in U_{\Delta, \Gamma}$. Furthermore, when z is near 0 and $t \in V_{\Delta, \text{areg}}^*$, the function $z \mapsto E_{\Gamma}(f)(t, z)$ defines an element of \mathcal{O}_{Δ} .*

If Γ is an admissible subset of N , we can take the value at t of the generalized function

$$E_{\Gamma}(f)(t, z) = \sum_{n \in \Gamma} e^{\langle t, z + 2i\pi n \rangle} f(z + 2i\pi n)$$

provided that t is in an alcove \mathfrak{a} . Thus, for $t \in \mathfrak{a}$, we can define the operator $E_{\Gamma}^t : R_{\Delta} \rightarrow \mathcal{O}_{\Delta}, f \mapsto E_{\Gamma}(f)(t, z)$. Now the argument of Theorem 19 proves

Proposition 25 *For $f \in G_{\Delta}$, $t \in V_{\Delta, \text{areg}}^*$ and $y \in U_{\Delta, \Gamma}$, we have*

$$E_{\Gamma}^t(f)(y) = \text{Tr}_{S_{\Delta}} \left(\text{Res}_{\Delta} m(e^{\langle t, \cdot \rangle} f) C\mathcal{T}(y) E_{\Gamma}^t \text{Res}_{\Delta} \right).$$

More explicitly, if we choose a diagonal basis OB then

$$E_{\Gamma}^t(f)(y) = \sum_{\sigma \in OB} \text{Res}^{\sigma\sigma} \left(f(z) e^{\langle t, z \rangle} F_{\Gamma, \sigma}^t(y - z) \right)$$

where $F_{\Gamma, \sigma}^t(z) = E_{\Gamma}(\phi_{\sigma})(t, z)$.

5 Witten series and Szenes formula

For $f \in R_\Delta$, let us form the series

$$Z(f)(t, z) = \sum_{n \in N_{reg}} e^{\langle t, z + 2i\pi n \rangle} f(z + 2i\pi n)$$

where N_{reg} is the set of regular elements of N . Then $Z(f)(t, z)$ is defined as a generalized function of t . As n varies in N_{reg} , this generalized function of t depends holomorphically on z when z varies in a neighborhood of 0. As N_{reg} is an admissible subset of N , we obtain from Lemma 24

Proposition 26 *For any alcove \mathfrak{a} , $Z(f)(t, z)$ is an analytic function of (t, z) when $t \in \mathfrak{a}$ and z is in a neighborhood of 0.*

We have

$$Z(f)(t, 0) = \sum_{n \in N_{reg}} e^{\langle t, 2i\pi n \rangle} f(2i\pi n).$$

This is well defined as a generalized function of t when t is in an alcove. If $n \mapsto f(2i\pi n)$ is sufficiently decreasing, then $Z(f)(t, 0)$ is a continuous function of t ; it generalizes the Bernoulli polynomial

$$B_k(t) = \sum_{n \neq 0} \frac{e^{2i\pi n t}}{(2i\pi n)^k}$$

where $0 < t < 1$.

We reformulate Szenes formula as an equality between $Z(f)(t, 0)$ and the constant term of the Eisenstein series $E(f)(t, z)$.

Theorem 27 *For any $f \in R_\Delta$ and t in an alcove \mathfrak{a} , we have*

$$Z(f)(t, 0) = CT(f)(t) = Tr_{S_\Delta} (Res_\Delta m(e^{\langle t, \cdot \rangle} f) C E^t Res_\Delta).$$

In particular, $Z(f)(t, 0)$ is a polynomial function of t when t varies in an alcove \mathfrak{a} .

As a consequence, if OB is a diagonal basis, then we recover the following residue formula (Theorem 4.4 of [3]):

$$\sum_{n \in N_{reg}} e^{\langle t, 2i\pi n \rangle} f(2i\pi n) = \sum_{o\sigma \in OB} Res^{o\sigma} (e^{\langle t, z \rangle} f(z) F_\sigma^{\mathfrak{a}}(-z)).$$

Thus, when

$$f = \frac{1}{\prod_{j=1}^k \alpha_j}$$

is sufficiently decreasing, this formula expresses the series

$$\sum_{n \in \mathbb{Z}^r, \langle \alpha_j, n \rangle \neq 0} \frac{1}{\prod_{j=1}^k \langle \alpha_j, 2i\pi n \rangle}$$

as an explicit rational number.

Proof. From the definitions of $Z(f)(t, z)$ and $CT(f)(t)$, we obtain for any $P \in S(V^*)$:

$$P(\partial_t)Z(f)(t, 0) = Z(Pf)(t, 0), \quad P(\partial_t)CT(f)(t) = CT(Pf)(t).$$

Thus, it is enough to prove that $Z(f)(t, 0) = CT(f)(t)$ for $f \in G_\Delta$, because G_Δ generates R_Δ as a $S(V^*)$ -module by Lemma 6.

For t in an alcove \mathfrak{a} , we can define the operator $Z^t : R_\Delta \rightarrow \mathcal{O}$ by

$$Z^t(f)(z) = \sum_{n \in N_{reg}} e^{\langle t, z + 2i\pi n \rangle} f(z + 2i\pi n).$$

The kernel formula holds for the operator Z^t . In particular, we obtain for $f \in G_\Delta$:

$$Z^t(f)(0) = Tr_{S_\Delta} (Res_\Delta m(e^{\langle t, \cdot \rangle} f) C Z^t Res_\Delta).$$

We thus need to prove that, for $f \in G_\Delta$,

$$Tr_{S_\Delta} (Res_\Delta m(e^{\langle t, \cdot \rangle} f) C (E^t - Z^t) Res_\Delta) = 0.$$

But E^t is given by a sum over the full lattice N , while Z^t is only over the regular elements of N . Thus, we can write (in many ways) $E^t - Z^t$ as a linear combination of operators $E_{\Gamma_\alpha}^t$ where each Γ_α is an admissible subset of N contained in the real hyperplane H_α . Now Szenes formula will follow from

Proposition 28 *Let Γ be an admissible subset of N contained in the real hyperplane H_α . Then, for $f \in G_\Delta$*

$$Tr_{S_\Delta} (Res_\Delta m(e^{\langle t, \cdot \rangle} f) C E_\Gamma^t Res_\Delta) = 0.$$

Proof. It suffices to prove that

$$\sum_{o\sigma \in OB} Res^{o\sigma} (e^{\langle t, z \rangle} f(z) E_{\Gamma}^t(\phi_{\sigma})(-z)) = 0$$

for some diagonal basis OB .

A total order on Δ provides us with a special diagonal basis OB of $OB(\Delta)$ (see for example [1], Proposition 14.) We choose this order such that α is minimal. In this case every element of OB is of the form $o\sigma = (\alpha_1, \alpha_2, \dots, \alpha_r)$ with $\alpha_1 = \alpha$. We claim that for each $o\sigma \in OB$,

$$Res^{o\sigma} (e^{\langle t, z \rangle} f(z) E_{\Gamma}^t(\phi_{\sigma})(-z)) = 0.$$

Indeed, we use the notation of Lemma 11 and write $V' = H_{\alpha}$. Then our set Γ is contained in V' . Thus

$$E_{\Gamma}^t(\phi_{\sigma})(z_1, z') = \frac{e^{t_1 z_1}}{z_1} \sum_{\gamma \in \Gamma} \frac{e^{\langle t', z' + 2i\pi\gamma \rangle}}{\prod_{j=2}^r \langle \alpha_j, z' + 2i\pi\gamma \rangle}.$$

We see that for t fixed and regular,

$$e^{\langle t, z \rangle} f(z) E_{\Gamma}^t(\phi_{\sigma})(-z) = \frac{1}{z_1} f(z_1, z') \psi(z')$$

where $f \in G_{\Delta}$ and $\psi(z')$ has poles at most on the complex hyperplanes $\alpha_j = 0$ for $j = 2, \dots, r$. Thus the claim follows from Lemma 11.

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